The Geometric Satake Equivalence

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Affine Grassmannians and its Geometry

Basics notations

In this report we fix following notations:

- k is an algebraic closed field.
- G is a reductive group over k with root datum $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$. The subset of dominant coweights is denoted as $X_\bullet(T)^+ \subset X_\bullet(T)$, and the subset of positive root is $\Phi^+ \subset \Phi$. Denote the sum of simple roots by $2\rho \in X^\bullet(T)$.
- Presheaves will means contravariant functor from the category of affine k-schemes to the category of sets.

Recall that a space E is called G-torsors on a scheme X if there is an action of G on E such that

(1) the map $E \to X$ is G-equivariant (where G acts trivially on X);

(2) for every $\operatorname{Spec} R \to X$, there is a faithfully flat map $\operatorname{Spec} R' \to \operatorname{Spec} R$ such that $\operatorname{Spec} R' \times_X E$ is G-equivariantly isomorphic to $\operatorname{Spec} R' \times G$.

On can roughly think E as a principal G-bundle on X.

Definition of Affine Grassmannians

Definition

The affine Grassmannian Gr_G of G is defined as following subscheme

$$\mathrm{Gr}_G(R) = \left\{ (\mathcal{E}, \beta) \colon \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } D_R, \\ \beta \colon \mathcal{E}|_{D_R^*} \xrightarrow{\sim} \mathcal{E}^0|_{D_R^*} \text{ is a trivialization} \end{array} \right\}.$$

Theorem

The presheaf Gr_G has a structure of ind-scheme[a]. If G is reductive, then Gr_G is ind-projective.

[a] An ind-scheme $X = \varinjlim X_i$ is a filtered limit of k-schemes with arrows $X_i \to X_j$ being closed embedding. We say X is ind-projective (or other property) if each X_i is projective (or such property).

Another interpretation of Gr_G

For a k-algebra R, We introduce following notations

$$D := \operatorname{Spec} k[\![t]\!], \quad D^* := \operatorname{Spec} k(\!(t)\!),$$
$$D_R := D \hat{\times} \operatorname{Spec} R := \operatorname{Spec} R[\![t]\!], \quad D_R^* := D^* \hat{\times} \operatorname{Spec} R := \operatorname{Spec} R(\!(t)\!).$$

Denote $\mathcal{O} = k[\![t]\!]$, $F = k(\!(t)\!)$. We also define following presheaf

$$\mathrm{Aut}(D)(R) = \mathrm{Aut}_R(R[\![t]\!]) \cong \operatorname{Spf} k[\![a_0]\!] \times \operatorname{Spec} k[a_1^{\pm 1}] \times \operatorname{Spec} k[a_2, \dots].$$

Another interpretation of Gr_G

Definition

Say X is a presheaf over $\mathcal{O}$. We define following presheaf

$$L^n X(R) := X(R[t]/(t^n)), \quad L^+ X(R) := X(R[\![t]\!]), \quad LX(R) := X(R(\!(t)\!)),$$

called the space of n-jets $L^n X$, the formal jet space $L^+ X$ and the loop space LX respectively. If X is a presheaf over k, we write $L^+ X(X \otimes_k \mathcal{O})$ by $L^+ X$ and $L(X \otimes_k F)$ by LX for simplicity.

For the case of affine Grassmannian, consider following action

$$LG \times \mathrm{Gr}_G \to \mathrm{Gr}_G, \quad (A, (\mathcal{E}, \beta)) \mapsto (\mathcal{E}, A\beta).$$

Proposition

The affine Grassmannian Gr_G can be identified with the fpqc quotient $[LG/L^+G]$.

Schubert Varieties

Fix an embedding $T \subset B \subset G$. For a coweight $\mu \in X_\bullet(T)$, it defines a map $\mu\colon F^\times \to T(F) \subset G(F)$, and the image of t is denoted by $t^\mu = \mu(t)$. Recall the Cartan decomposition

$$G(F) = \bigsqcup_{\mu \in X_\bullet(T)^+} G(\mathcal{O}) t^\mu G(\mathcal{O})$$

which implies following isomorphism (since $G(\mathcal{O}) t^\mu G(\mathcal{O})$ is stable under a change of uniformizer t)

$$G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}) \cong X_\bullet(T)/W \cong X_\bullet(T)^+.$$

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Now let $\mathcal{E}_1, \mathcal{E}_2$ be two G-torsors over D and

$$\beta\colon \mathcal{E}_1|_{D^*} \xrightarrow{\sim} \mathcal{E}_2|_{D^*}$$

be an isomorphism over D^*. By choosing isomorphisms $\phi_1\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0$ and $\phi_2\colon \mathcal{E}_2 \xrightarrow{\sim} \mathcal{E}^0$, one obtains an automorphism of the trivial G-torsor $\phi_2 \beta \phi_1^{-1} \in \mathrm{Aut}(\mathcal{E}^0|_{D^*})$, whence an element in $G(F)$.

Schubert Varieties

Different choices of ϕ_1 and ϕ_2 will modify this element by left and right multiplication by elements from $G(\mathcal{O})$. Therefore, β gives rise to a well-defined element

$$\mathrm{Inv}(\beta) \in G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$$

Now for two G-torsors $\mathcal{E}_1, \mathcal{E}_2$ over D_R and $\beta\colon \mathcal{E}_1|_{D_R^*} \xrightarrow{\sim} \mathcal{E}_2|_{D_R^*}$, each $x \in \operatorname{Spec} R$ defines $\mathrm{Inv}_x(\beta) := \mathrm{Inv}(\beta_{k(x)}) \in \mathrm{Aut}(\mathcal{E}^0|_{D_{k(x)}^*})$ by base changing.

Proposition

Let $X = \operatorname{Spec} R$, then for $\mu \in X_\bullet(T)^+$, the set

$$X_{\leq\mu} := \{x \in X : \mathrm{Inv}_x(\beta) \leq \mu\}$$

is Zariski closed in X.

Schubert Varieties

Now define the Schubert variety of Gr_G as the closed subset

$$\mathrm{Gr}_{\leq\mu} := \{(\mathcal{E}, \beta) \in \mathrm{Gr} \colon \mathrm{Inv}(\beta) \leq \mu\}$$

with reduced scheme structure, and the Schubert cell

$$\mathrm{Gr}_{\mu} := \{(\mathcal{E}, \beta) \in \mathrm{Gr} \colon \mathrm{Inv}(\beta) = \mu\} = \mathrm{Gr}_{\leq\mu} \setminus \bigcup_{\lambda<\mu} \mathrm{Gr}_{\leq\lambda}$$

which is an open subset of $\mathrm{Gr}_{\leq\mu}$.

Proposition

(1) Gr_{μ} *is a single* $L^{+}G$*-orbit and is a smooth quasi-projective variety with dimension* $(2\rho, \mu)$.

(2) $\mathrm{Gr}_{\leq\mu}$ *is the Zariski closure of* Gr_{μ}*, whence a projective variety.*

Sketch.

For (1), note that the stabilizer $\mathrm{Stab}_{L^+G}(t^\mu) = L^+G \cap t^\mu L^+G t^{-\mu}$, which implies the map

$$L^+G/(L^+G \cap t^\mu L^+G t^{-\mu}) \to LG/L^+G, \quad g \mapsto gt^\mu$$

is a locally closed embedding with image Gr_μ. Therefore, Gr_μ is smooth, and its dimension can be calculated by its tangent space at t^μ.
For (2), note that if $\lambda \leq \mu$, then there is a positive coroot α such that $\mu - \alpha$ is dominant and $\lambda \leq \mu - \alpha \leq \mu$. Hence it is enough to show that $t^{\mu-\alpha}$ is in the Zariski closure of Gr_μ, which is achieved by constructing a curve C in $\mathrm{Gr}_{\leq\mu}$ such that $t^{\mu-\alpha} \in C$ and $C \setminus \{t^{\mu-\alpha}\} \subset \mathrm{Gr}_\mu$. $\square$

Opposite Schubert Varieties

Let $L^-G(R) = G(R[t^{-1}])$. One has Birkhoff decomposition

$$G(F) = \bigsqcup_{\mu \in X_\bullet(T)^+} G(k[t^{-1}])t^\mu G(\mathcal{O})$$

which induces isomorphism $L^-G(k)\backslash G(F)/G(\mathcal{O}) \cong X_\bullet(T)^+$. We define opposite Schubert cells as L^-G-orbits on G, and opposite Schubert "variety" as follows

$$\mathrm{Gr}^\mu = L^-G \cdot t^\mu \subset \mathrm{Gr}, \quad \mathrm{Gr}^{\geq \mu} = \bigsqcup_{\lambda \geq \mu} \mathrm{Gr}^\lambda.$$

Proposition

(1) Gr^μ *is a locally closed sub-ind-scheme of* Gr.
(2) $\mathrm{Gr}_\mu \cap \mathrm{Gr}^\lambda \neq \varnothing$ *iff* $\mu \geq \lambda$. *And* $\mathrm{Gr}_\mu \cap \mathrm{Gr}^\mu = G \cdot t^\mu \cong G/P_\mu$.
(3) $\mathrm{Gr}^{\geq\mu}$ *is Zariski closed, and contains* Gr^μ *as an open dense subset.*
(4) The codimension of $\mathrm{Gr}^{\geq\mu}$ *is* $(2\rho, \mu) - \dim G/P_\mu$.

The Satake Category

Perverse Sheaves on Ind-scheme

Let $X = \varinjlim_i X_i$ be an ind-scheme. We can define the category of perverse sheaves on X as $\mathsf{Perv}(X) = \varinjlim \mathsf{Perv}(X_i)$. Now say X is acted by a pro-algebraic group K. We assume that

(S) The stabilizer of each geometric point only has finitely many connected components.

Then we can define the category of K-equivariant perverse sheaves on X (the detailed construction has to be omitted due to time constraint).

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Then we can define the category of K-equivariant perverse sheaves on X (the detailed construction has to be omitted due to time constraint).
In particular, the action of L^+G on Gr satisfies the condition (S) above, since each orbit is of the form $\mathrm{Gr}_\mu = L^+G/(L^+G \cap t^\mu L^+Gt^{-\mu})$, and $L^+G \cap t^\mu L^+Gt^{-\mu}$ is connected.

The Satake Category

Now we define the Satake category Sat_G as the category $\mathsf{P}_{L^+G}(\mathrm{Gr})$ of L^+G-equivariant perverse sheaves on Gr. Denote the intersection complex of $\mathrm{Gr}_{\leq\mu}$ by IC_μ. Then $\mathrm{IC}_\mu|_{\mathrm{Gr}_\mu} = \overline{\mathbb{Q}}_\ell[(2\rho, \mu)]$, and its restriction to each stratum Gr_λ is constant. Again since $L^+G \cap t^\mu L^+G t^{-\mu}$ is connected, irreducible objects of $\mathsf{Perv}_{L^+G}(\mathrm{Gr})$ are exactly these IC_μ's.

Proposition

The category $\mathsf{P}_{L^+G}(\mathrm{Gr})$ is semisimple.

Sketch.

We only need to show $\mathrm{Ext}^1(\mathrm{IC}_\mu, \mathrm{IC}_\lambda) = 0$ for any λ, μ. The non-trivial case is $\lambda \leq \mu$ or $\mu \leq \lambda$, which follows from two facts.

(i) Purity of the stalk cohomology of IC_μ.

(ii) $p(\mathrm{Gr}_\mu) = (-1)^{\dim \mathrm{Gr}_\mu}$, where $p\colon \pi_0(\mathrm{Gr}_G) \to \mathbb{Z}/2$ is induced by $X_\bullet(T) \to \mathbb{Z}/2, \mu \mapsto (-1)^{(2\rho,\mu)}$.

□

The Main Theorem

Recall that a root datum $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$ determines a connected reductive group (over an algebraic closed field) up to isomorphism. For such group G with root datum $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$, one can define its Langland's dual group that is determined by the dual root datum $(X_\bullet, \Phi^\vee, X^\bullet, \Phi)$.

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Theorem

There is an equivalence of category

$$\mathrm{Sat}_G \xrightarrow{\sim} \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$$

where $\hat{G}$ is the Langland's dual group of G.

The equivalence is actually of symmetric monoidal category, and the monoidal structure on Sat_G will be define now.

The Symmetric Monoidal Structure on the Satake Category

The Convolution Product I

$\mathrm{Sat}_G = \mathsf{Perv}(L^+G \setminus LG/L^+G)$ admits a natural monoidal structure:

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} LG \overset{L^+G}{\times} \mathrm{Gr}_G =: \mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G,$$

here m is the action map, p is $L^+G \times L^+G$ equivariant and q is a L^+G-torsor. This gives

$$D_{L^+G \times L^+G}(\mathrm{Gr}_G \times \mathrm{Gr}_G) \xrightarrow{p^*} D_{L^+G \times L^+G}(LG \times \mathrm{Gr}_G) \xrightarrow{(q^*)^{-1}} D_{L^+G}(LG \overset{L^+G}{\times} \mathrm{Gr}_G) \xrightarrow{m_!=m_*} D_{L^+G}(\mathrm{Gr}_G),$$

For $\mathcal{F}, \mathcal{G} \in D_{L^+G}(\mathrm{Gr}_G)$, we define the **twisted external product**

$$\mathcal{F} \widetilde{\boxtimes} \mathcal{G} := (q^*)^{-1} p^* (\mathcal{F} \boxtimes \mathcal{G}),$$

and the **convolution product**

$$\mathcal{F} \star \mathcal{G} := m_! (\mathcal{F} \widetilde{\boxtimes} \mathcal{G}).$$

FACTS:

The Convolution Product II

(1) m is semismall, hence Sat_G is closed under convolution;
(2) $(\mathrm{Sat}_G, \star, \mathsf{IC}_0)$ is a monoidal category.

The main goal of this part is to give a sketch of:

Theorem

$(\mathrm{Sat}, \star, \mathsf{IC}_0)$ admits a **symmetric** monoidal structure, s.t. taking hypercohomology $\mathrm{H}^*\colon \mathrm{Sat}_G \to \mathbf{Vect}$ is symmetric monoidal.

We want to define a "global" version of Gr_G. Let X be a smooth connected curve. For every finite set I, we define

$$L^+G_{X^I}\colon R \mapsto \{(x,\beta) : x \in X^I(R), \beta \in G(\hat{\Gamma}_x)\},$$

$$LG_{X^I}\colon R \mapsto \{(x,\beta) : x \in X^I(R), \beta \in G(\hat{\Gamma}^\circ_x)\},$$

$$\mathrm{Gr}_{G,X^I} = LG_{X^I}/L^+G_{X^I}\colon R \mapsto$$

$$\{(x,\mathcal{E},\beta) : x \in X^I(R),\ \mathcal{E} \in \mathsf{Bun}_G(X)(R),\ \beta\colon \mathcal{E}|_{X_R\setminus\Gamma_x} \simeq \mathcal{E}^0|_{X_R\setminus\Gamma_x}\}.$$

Here, $\Gamma_x \subset X_R$ is the graph of $x \in X^I(R)$, $\hat{\Gamma}_x$ is the formal completion of Γ_x in X_R which is an ind-affine scheme and $\hat{\Gamma}^\circ_x := \mathrm{Spec}\,\mathcal{O}_{\hat{\Gamma}_x}(\hat{\Gamma}_x) \setminus \Gamma_x$.
FACTS:

(1) $L^+G_{X^I}$ is represented by a scheme affine over X^I;
(2) $L^G_{X^I}$ is represented by an ind-scheme over X^I;

Beilinson–Drinfeld Grassmannian II

(3) Gr_{G,X^I} is represented by an ind-scheme, ind-of finite type over X^I, and ind-projective if G is reductive.

(4) Define

$$\hat{X}\colon R \mapsto \{(x,\alpha) : x \in X(R), \alpha\colon \mathcal{O}_{\hat{\Gamma}_x}(\hat{\Gamma}_x) \simeq R[[t]]\} \to X.$$

This is an $\mathrm{Aut}(D)$-torsor. We have $\mathrm{Gr}_{G,X} \simeq \hat{X} \overset{\mathrm{Aut}(D)}{\times} \mathrm{Gr}_G$.

Convolution Grassmannian I

We want to define the global version of the convolution product. For a surjection $\alpha\colon J \twoheadrightarrow \{1,\ldots,n\}$ between finite sets, we define

$$\mathrm{Gr}_{G,\alpha}\colon R \mapsto$$

$$\{(x,\mathcal{E}^i,\beta_i) : x \in X^J(R),\ \mathcal{E}_i \in \mathsf{Bun}_G(X)(R), \beta_i\colon \mathcal{E}_i|_{X_R\setminus\Gamma_{x_i}} \simeq \mathcal{E}_{i-1}|_{X_R\setminus\Gamma_{x_i}}\},$$

where $x_i = x|_{J_i:=\alpha^{-1}(i)}$.
FACT: Using some suitable torsors, one can show that

$$\mathrm{Gr}_{G,\alpha} \simeq \mathrm{Gr}_{G,X^{J_1}} \widetilde{\times} \mathrm{Gr}_{G,X^{J_2}} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{G,X^{J_n}}.$$

We define $m_\alpha\colon \mathrm{Gr}_{G,\alpha} \to \mathrm{Gr}_{G,X^J}$ by sending $(x,\mathcal{E}_i,\beta_i)$ to $(x,\mathcal{E}_n,\beta_1\cdots\beta_n)$. Define $X^{(\alpha)} \subset X^J$ to be the open locus of those $(x_j)_{j\in J}$ with $x_j \neq x_{j'}$ for all $\alpha(j) \neq \alpha(j')$. It is not hard to show that $m_\alpha|_{X^{(\alpha)}}$ is an isomorphism.

Convolution Grassmannian II

For $\mathcal{F}_i \in D_{L^+G_{X^I}}(\mathrm{Gr}_{G,X^{J_i}})$, we define the **external convolution product** to be

$$\mathcal{F}_1 \bigstar \cdots \bigstar \mathcal{F}_n := m_{\alpha,!}(\mathcal{F}_1 \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} \mathcal{F}_n) \in D_{L^+G_{X^J}}(\mathrm{Gr}_{G,X^J}).$$

Factorization Structures I

By looking at points, one can easily show that

$$\mathrm{Gr}_{G,X^2} \times_{X^2} \Delta \simeq \mathrm{Gr}_{G,X},$$

$$\mathrm{Gr}_{G,X^2}|_{X^2 \setminus \Delta} \simeq (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_{X^2 \setminus \Delta}.$$

More generally, given a surjective map $\alpha \colon J \twoheadrightarrow I$, let $\Delta_\alpha \colon X^I \to X^J$ be the corresponding diagonal, then

$$\Delta_\alpha \colon \mathrm{Gr}_{G,X^I} \xrightarrow{\simeq} \mathrm{Gr}_{G,X^J} \times_{X^J} X^I.$$

Also, one can construct an isomorphism

$$c_\alpha \colon \mathrm{Gr}_{G,X^J}|_{X^{(\alpha)}} \simeq \left(\prod_{i \in I} \mathrm{Gr}_{G,X^{J_i}}\right)\Big|_{X^{(\alpha)}}.$$

In summary, Gr_{G,X^I} together with the maps Δ_α, c_α and the unit $X^I \to \mathrm{Gr}_{G,X^I}$ given by the trivial torsor forms a **unital factorization space** over X.

Monoidality I

Define $\mathrm{Sat}_{G,X^I} \subset \mathsf{Perv}_{L^+G_{X^I}}(\mathrm{Gr}_{G,X^I})$ to be the subcategory of objects that are ULA (universally locally acyclic) w.r.t. $q_I \colon \mathrm{Gr}_{G,X^I} \to X^I$.
FACTS:

(1) For every surjection $\alpha \colon J \twoheadrightarrow I$ between finite sets, the functor $\Delta_\alpha^![|J| - |I|] = \Delta_\alpha^*[|I| - |J|]$ preserves ULA sheaves, hence gives a functor $\Delta_\alpha \colon \mathrm{Sat}_{G,X^J} \to \mathrm{Sat}_{G,X^I}$;

(2) For every $\mathcal{F}_i \in \mathrm{Sat}_{G,X^{J_i}}$, the external convolution

$$\mathcal{F}_1 \bigstar \cdots \bigstar \mathcal{F}_n = j_{!*}(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n) \in \mathrm{Sat}_{G,X^J}.$$

By these two facts, we can define the internal convolution $\star$ as the Δ-pullback of external convolution.

The isomorphism $\mathrm{Gr}_X \simeq \hat{X} \overset{\mathrm{Aut}(D)}{\times} \mathrm{Gr}_G$ gives a functor $\mathrm{Sat}_G \to \mathrm{Sat}_{G,X}$ by sending $\mathcal{F}$ to $1 \widetilde{\boxtimes} \mathcal{F}$.
FACT: This functor is fully faithful. (Essentially because X is smooth.)

Monoidality II

The hypercohomology $\mathrm{H}^*\colon \mathrm{Sat}_G \to \mathbf{Vect}$ factors as follows: For every $\mathcal{F} \in \mathrm{Sat}_{G,X^I}$, since q_I is ind-proper, $q_{I,*}\mathcal{F} \in D(X^I)$ is also ULA. Hence $\mathcal{H}^i(q_{I,*}\mathcal{F})$ is a local system by the smoothness of X. Define

$$\mathcal{H}(\mathcal{F}) := \bigoplus_i \mathcal{H}^i(q_{I,*}\mathcal{F})[i].$$

FACT: $\mathcal{H}\colon \mathrm{Sat}_{G,X} \to \mathrm{Loc}^{\mathrm{gr}}(X)$ is monoidal and for every closed point $x \in X$, the stalk of $\mathcal{H}(\mathcal{F})$ at x is $\mathrm{H}^*\,\mathcal{F}$ for all $\mathcal{F} \in \mathrm{Sat}_G$. (The monoidality follows from the Künneth formula and the formula for external convolution.)

Braidings I

Finally, we construct the braiding on $\mathrm{Sat}_{G,X}$.
Note that there is an involution $\sigma\colon \mathrm{Gr}_{G,X^2} \to \mathrm{Gr}_{G,X^2}$ flipping X^2. Using the explicit formula for external convolutions, one can show

$$\sigma^*(\mathcal{F}_1 \bigstar \mathcal{F}_2) \simeq (\mathcal{F}_2 \bigstar \mathcal{F}_1).$$

Composite with $\Delta_{\{1,2\}\to\{1\}}$, this gives $\mathcal{F}_1 \star \mathcal{F}_2 \simeq \mathcal{F}_2 \star \mathcal{F}_1$.
By direct computation, one can show that $\mathcal{H}$ is actually symmetric monoidal.
Since Sat_G is a full subcategory of Sat_X, H^* is symmetric monoidal.

The identification of the dual group

Now we have a functor $H^* : \mathrm{Sat}_G \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$, which is exact, faithful, $\overline{\mathbb{Q}}_\ell$-linear, and symmetric monoidal. This allows us to apply the so-called Tannakian formalism.

Definition

Let $\mathcal{C}$ be an rigid Abelian tensor category such that $\mathrm{End}(1_{\mathcal{C}}) = k$. We define a fiber functor for $\mathcal{C}$ to be an exact, faithful, k-linear, and symmetric monoidal functor $\omega : \mathcal{C} \to \mathbf{Vect}_k$.

Tannakian formalism

Theorem

For a fiber functor $\omega : \mathcal{C} \to \mathbf{Vect}_k$, we have a symmetric monoidal equivalence of Abelian categories $\mathcal{C} \to \mathbf{Rep}(G)$, where G is the affine group scheme over k, such that $G(R) = \mathrm{Aut}^{\otimes}(\omega \otimes_k R)$, where we have

$$\omega \otimes_k R : \mathcal{C} \to \mathbf{Mod}_R, c \mapsto \omega(c) \otimes_k R$$

In this case, the category $\mathcal{C}$ is called a Tannakian category.
Moreover, if there is no non-trivial representation X such that the subquotients of X^n is stable under $\otimes$, then G is connected. If moreover $\mathcal{C}$ is semisimple with a tensor generator, we have G is connected reductive.

The case of a torus

Definition

Consider $H^* : \mathrm{Sat}_G \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$. We let $\tilde{G} = \mathrm{Aut}^{\otimes}(H^*)$

So now it suffices to identify $\tilde{G}$ with $\hat{G}$!

The case of a torus

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Lemma

For T a torus, we have $\tilde{T} = \hat{T}$.

Proof.

In this case, $Gr_T = \amalg_\lambda t^\lambda$, so $\mathrm{Sat}_G = X_\bullet(T)\text{-gr}\mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$, and the fiber functor H^* can be identified with the forgetful functor removing the gradings. $\square$

Parabolic induction functor

Now, it suffices to reduce the general case to the case of a torus. As usual, we use the parabolic induction.

Construction

We fix a Borel $i : B \subset G$ and an Abstract Cartan $q : B \twoheadrightarrow T$. The maps induces

$$\mathrm{Gr}_T \xleftarrow{q} \mathrm{Gr}_B \xrightarrow{i} \mathrm{Gr}_G$$

For $\lambda \in X_\bullet(T)$, we denote $S_\lambda = i(q^{-1}(t^\lambda))$.

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The main lemma we will use is that

Theorem

We have an isomorphism of functors

$$H^*(-) \simeq \bigoplus_{\lambda \in X_\bullet(T)} H^*(S_\lambda, -)$$

Parabolic induction functor

proof

We sketch the proof.
Firstly, notice that $H_c^*(S_\lambda, \mathcal{A})$ is concentrated in degree $(2\rho, \lambda)$, with the cohomology

$$H_c^{(2\rho,\lambda)}(S_\lambda, IC_\mu)((\rho, \lambda+\mu)) \simeq \overline{\mathbb{Q}}_\ell[\mathrm{Irr}(S_\lambda \cap \mathrm{Gr}_{\leq\mu})]$$

Then, note that the stratification $\{S_\lambda\}$ of Gr_G gives a complete filtration on $H^*(-)$ by

$$\mathrm{Fil}_{\geq\mu} H^*(-) = \ker(H^*(-) \to H^*(S_{<\lambda}, -))$$

which, by taking graded pieces, gives the desired decomposition

$$H^* = \bigoplus H_c^*(S_\lambda, -)$$

Categorical Satake transform

Definition

We let the categorical Satake transform functor $\mathrm{CT} : \mathrm{Sat}_G \to \mathrm{Sat}_T$ denote $\lambda \mapsto H_c^*(S_\lambda, -)$, which maps a perverse sheaf to a $X_\bullet(T)$-graded vector space.
By definition, we have $H^* \circ \mathrm{CT} = H^* : \mathrm{Sat}_G \to \mathrm{Sat}_T \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$.

The functor CT is actually monoidal! This gives a functor between the Tannakian categories $\mathrm{Sat}_G \to \mathrm{Sat}_T$, so an induced homomorphism $\tilde{T} \to \tilde{G}$. Moreover, it is maximal by the properties of the IC sheaves. Now, it suffices to identify the roots datum via the Tannakian formalism.

Categorical Satake transform

proof

We now prove the main theorem. Notice that fixing a Borel subgroup in $\tilde{G}$ amounts to fixing a highest weight subspace in every irreducible representation of $\tilde{G}$ (so as giving an ordering on the set of weights).
For $\mu \in X_\bullet(T)^+$ a dominant coweight, let

$$\ell_\mu = H^*(2\rho, \mu)_c(S_\mu, IC_\mu) \subset H_c^*(IC_\mu) = L_\mu$$

Notice that the natural map $L_{\mu_1} \otimes L_{\mu_2} \to L_{\mu_1+\mu_2}$ sends the tensor $\ell_{\mu_1} \otimes \ell_{\mu_2}$ to $\ell_{\mu_1+\mu_2}$, so $\ell_\mu \subset L_\mu$ fixes a Borel $\tilde{B}$. Hence for this $\tilde{B}$, we have the dominant weights $X^\bullet(\tilde{T})^+$ identified with $X_\bullet(T)^+$.
Then recall that the positive roots can be characterized via the representations by the following process: λ is positive iff there for some μ, we have $\mu - \lambda$ a weight of L_λ. By the characterization of the Satake transform functor, this is equivalent to $t^{\mu-\lambda} \in \mathrm{Gr}_{\leq\mu}$, hence the positive (hence simple) coroots of $\tilde{G}$ corresponds to the positive roots of G.